On the dynamics of the convolution of probabilities in topological groups

A. Baraviera, E. R. Oliveira and F. B. Rodrigues

April 29, 2013

Abstract

In this work we are going to study the main dynamical properties of the dynamical system $\Phi_\nu : \mathcal{P}(G) \to \mathcal{P}(G)$, where G is a compact topological group and $\Phi_\nu(\mu) = \nu * \mu$.

1 Introduction

Given a topological group G and a probability measure $\nu \in \mathcal{P}(G)$ there exists a natural dynamical system defined on $\mathcal{P}(G)$, that is called the convolution and maps the probability μ to $\nu * \mu$. For example, when G is the additive group $\mathbb{R}$, ν and μ are the probability distributions of the real random variables X and Y this operation can be seen as the procedure that gives the probability distribution of the random variable $X + Y$; hence, in order to understand the iterated sum of random variables with the same distribution one can, for example, try to understand the limits (or at least accumulation points) of the convolution dynamics on the probability space $\mathcal{P}(G)$.

Motivated by this application and also by the study of another dynamical system on a probability space we use some dynamical ideas in order to describe the limit points of the orbits; since compactness is an important tool in what follows we impose from the beginning that the group G is compact, but the example above mentioned with the group $\mathbb{R}$ shows that some of the reasonings explored here could, perhaps, be extended to the non compact setting, possibly with some extra hypothesis.

The problem of study powers of convolution of probability measures has been studied in several papers in the last few years and has several applications in statistics and group theory see [6].

For instance, in [12] the authors develops cyclic conditions to weak convergence of powers of convolutions over bi-stochastic matrices.

Our goal is to put the powers of convolution as iterates of a dynamical system, so its the behavior can be understood from the dynamical properties and from the knowledge of other dynamical systems on spaces of probabilities as in [1] and [2].

We hope that in the future we will be able to extract a lot of important connections between dynamics and convergence of powers of convolutions.

This paper is structured as follows. In the first part, where we developed the theory for any compact topological group, we define the dynamic and study some basic topological properties of the convolution map like continuity of the convolution map, topological entropy, the ω-limit set of a measure $\mu \in \mathcal{P}(G)$. One of the main results of this part is

Theorem 10 *Let $\sim$ be a relation on $\mathcal{P}(G)$ given by*

$$\nu_1 \sim \nu_2 \Longleftrightarrow \exists \varphi \in \mathrm{Aut(G)} \ s.t. \ \varphi_\sharp(\nu_1) = \nu_2.$$

Then $\sim$ is an equivalence relation and, for each $\nu \in \mathcal{P}(G)$ the set

$$\{\Phi_{\nu'} \mid \nu' \in [\nu] \in \mathcal{P}(G)/\sim\},$$

is such that any two systems Φ_{ν_1} and Φ_{ν_2} are topologically conjugated.

The above theorem give us a kind of topological classification. The notation used in it will become clear throughout the text.

In the second part we consider the convolution map when G is a finite group. There we could see that the convolution map is associated to a doubly stochastic matrix, and using the theory developed for this type of matrices we obtained a complete description of the ω-limit set of a measure when the support and the group generated by the support of the measure ν satisfy certain conditions. One of the main results of this part is

Theorem 44 *Let $G = \{g_0, ..., g_{n-1}\}$ a finite group, $\nu = p \in \mathcal{P}(G)$ a acyclic probability and $H = \langle Z_+(p)\rangle$. The matrix $p(G^{-1} \times G) = (p(g_i^{-1}g_j))_{i,j}$ satisfies $\lim_{k\to\infty} p(G^{-1} \times G)^k = B$, where B is the matrix given by*

$$b_{ij} = \begin{cases} 0, & \text{if } g_i^{-1}g_j \notin H \\ \frac{1}{|H|}, & \text{if } g_i^{-1}g_j \in H \end{cases} .$$

2 First Properties

Here we give the first definitions and prove some basic properties of the convolution map. As we are interested in dynamical properties, we just remind that the convolution map is continuous in our setting.

Let $(G, \cdot)$ a compact topological group. In the next, we will omit the symbol $\cdot$ in the group operations in order to simplify the notation: for example we will write $g_1 \cdot g_2 = g_1 g_2$. Let $\mathcal{P}(G)$ be the set of probability measures on G. Given $\mu, \nu \in \mathcal{P}(G)$ we define the weak-$*$ distance in $\mathcal{P}(G)$ given by

$$d(\nu, \mu) = \sum_{i=1}^{\infty} \frac{1}{2^i} \Big| \int_G f_i(g) d\mu(g) - \int_G f_i(g) d\nu(g) \Big|,$$

where $f_i : G \to [0,1]$ is a dense sequence in $C(G, [0,1])$. The weak-$*$ distance is a metric and generates the weak topology in $\mathcal{P}(G)$, and if G is a compact metric space, then $\mathcal{P}(G)$ with the weak-$*$ distance is also a compact metric space Let $\nu \in \mathcal{P}(G)$. We define the convolution map $\Phi_\nu : \mathcal{P}(G) \to \mathcal{P}(G)$ by $\Phi_\nu(\mu) = \nu * \mu$, where

$$\nu * \mu(E) = \int_G \nu(Ey^{-1}) d\mu(y).$$

Approximating by simple functions we easily get the property for integrals.

Lemma 1. *Let $f \in C(G, \mathbb{R})$ and $\mu \in \mathcal{P}(G)$. Then*

$$\int f d(\Phi_\nu(\mu)) = \int_G f d\nu * \mu = \int_G \int_G f(xy) d\nu(x) d\mu(y).$$

The proof of the continuity can be found in [10]:

Lemma 2. *The map $\Phi_\nu : \mathcal{P}(G) \to \mathcal{P}(G)$, where*

$$\Phi_\nu(\mu)(E) = \nu * \mu(E),$$

is continuous in the weak topology.

There is a relation between the convolution map and the push forward of a measure. In order to see that we recall the definition of push forward.

Let X a connected compact separable metric space. If we consider a continuous map $T \colon X \to X$ it induces a map

$$T_\sharp : \mathcal{P}(X) \to \mathcal{P}(X),$$

where $T_\sharp(\mu)(A) = \mu(T^{-1}(A))$. This map is called *the push forward of T*. Additional dynamical properties of this map can be founded in [1] and [2].

Lemma 3. *If $g \in G$, then $\Phi_\nu(\delta_g) = (R_g)_\sharp \nu$, where R_g right translation defined by g.*

Proof. Take $g \in G$ and notice that

$$\begin{aligned}\Phi_\nu(\delta_g)(E) &= \int_G \int_G \chi_E(xy) d\nu(x) d\delta_g(y) = \int_G \chi_E(xg) d\nu(x) \\ &= \int_G \chi_E(x) d((R_g)_\sharp \nu)(x) = (R_g)_\sharp \nu(E)\end{aligned}$$

□

Theorem 4. *Given $\nu \in \mathcal{P}(G)$, the map $\Phi_\nu : \mathcal{P}(G) \to \mathcal{P}(G)$ has a fixed point.*

Proof. We know that $\mathcal{P}(G)$ is a compact convex set and Φ_ν is continuous. By *The Schauder Fixed Point Theorem* we have that Φ_ν has a fixed point. □

Remark 5. *By Theorem 4 we see that always there exists a probability measure $\mu \in \mathcal{P}(G)$ such that $\nu * \mu = \mu$.*

Since we have a continuous map, we would to know about the topological dynamics defined by that map. The first step in this direction is to understand the orbits of the map Φ_ν.

Lemma 6. *Given $\mu \in \mathcal{P}(G)$, the orbit of μ by the map Φ_ν is given by*

$$\mathcal{O}(\mu) = \{\nu^n * \mu : \nu^n = \underbrace{\nu * \nu * ... * \nu}_{n},\ n \in \mathbb{N}\}.$$

Proof. We notice that

$$\begin{aligned}\Phi_\nu^2(\mu)(E) = (\nu * (\nu * \mu))(E) &= \int_G \int_G \chi_E(xy) d\nu(x) d(\nu * \mu)(y) \\ &= \int_G \int_G \int_G \chi_E(xyz) d\nu(x) d\nu(z) d\mu(x).\end{aligned}$$

By the other hand we have

$$\begin{aligned}(\nu * \nu) * \mu(E) &= \int_G \int_G \chi_E(xy) d(\nu * \nu)(x) d\mu(y) \\ &= \int_G \int_G \int_G \chi_E(xyz) d\nu(x) d\nu(z) d\mu(x).\end{aligned}$$

Then we get $\Phi_\nu^2(\mu) = (\nu*\nu)*\mu = \nu^2*\mu$. By induction we get the result. □

2.1 Topological classification

Let $(G_1, \Box)$ and $(G_2, \circ)$ be two topological compact groups and $\varphi : G_1 \to G_2$ an isomorphism between G_1 and G_2. To make easier the notation for $x, y \in G_1$, we write xy instead of $x \Box y$, and for $g, h \in G_2$ we write gh instead of $g \circ h$.

Lemma 7. *Let $\varphi_\sharp : \mathcal{P}(G_1) \to \mathcal{P}(G_2)$ be the push forward of the isomorphism $\varphi : G_1 \to G_2$, then*

$$\varphi_\sharp(\nu * \mu) = \varphi_\sharp(\nu) * \varphi_\sharp(\mu), \text{ for all } \nu, \mu \in \mathcal{P}(G_1).$$

Proof. Consider $\nu, \mu \in \mathcal{P}(G_1)$, $f \in C(G_2, \mathbb{R})$, then

$$\begin{aligned}\varphi_\sharp(\nu * \mu)(f) &= \int_{G_2} f(g) d\varphi_\sharp(\nu * \mu)(g) = \int_{G_1} (f \circ \varphi)(t) d(\nu * \mu)(t) \\ &= \int_{G_1}\int_{G_1} (f \circ \varphi)(xy) d\nu(x) d\mu(y) \\ &= \int_{G_2}\int_{G_2} f(gh) d\varphi_\sharp(\nu)(g) d\varphi_\sharp(\mu)(h) \\ &= (\varphi_\sharp(\nu) * \varphi_\sharp(\mu))(f), \quad \forall f \in C(G_2, \mathbb{R}).\end{aligned}$$

Thus $\varphi_\sharp(\nu * \mu) = \varphi_\sharp(\nu) * \varphi_\sharp(\mu)$. □

Given $\nu_1 \in \mathcal{P}(G_1)$ we take $\nu_2 = \varphi_\sharp(\nu_1) \in \mathcal{P}(G_2)$. We consider the maps

$$\Phi_{\nu_1} : \mathcal{P}(G_1) \to \mathcal{P}(G_1) \text{ and } \Phi_{\nu_2} : \mathcal{P}(G_2) \to \mathcal{P}(G_2).$$

Then we define a map

$$\begin{aligned} H : \mathcal{P}(G_1) &\to \mathcal{P}(G_2) \\ \mu &\mapsto \varphi_\sharp \mu. \end{aligned}$$

Lemma 8. *The map $H : \mathcal{P}(G_1) \to \mathcal{P}(G_2)$, where $H(\mu) = \varphi_\sharp \mu$ is a homeomorphism.*

Proof. As φ is an isomorphism of topological groups we have that its push forward is a continuous bijection. As $\mathcal{P}(G)$ is a compact space we have that the inverse of a continuous map is a continuous map and we the result. □

The push forward of an isomorphism plays an important role giving a topological conjugation criteria.

Proposition 9. *Let $\varphi : G_1 \to G_2$ be an isomorphism between topological groups. Take $\nu_1 \in \mathcal{P}(G_1)$, $\nu_2 \in \mathcal{P}(G_2)$ such that $\varphi_\sharp(\nu_1) = \nu_2$, then we have that the maps $\Phi_{\nu_i} : \mathcal{P}(G_i) \to \mathcal{P}(G_i)$ for $i = 1, 2$ are topologically conjugated.*

Proof. Take H as in Lemma 8. For any $\mu \in \mathcal{P}(G_1)$ we notice that

$$H \circ \Phi_{\nu_1}(\mu) = \varphi_\sharp(\nu_1 * \mu) = \varphi_\sharp(\nu_1) * \varphi_\sharp(\mu) = \nu_2 * \varphi_\sharp(\mu) = \Phi_{\nu_2} \circ H(\mu),$$

so Φ_{ν_1} and Φ_{ν_2} are topologically conjugated. □

For a topological group $(G, \cdot)$ we define the set of automorphisms of G as being

$$\mathrm{Aut}(G) = \{\varphi : G \to G \mid \text{ continuous isomorphism with continuous inverse}\},$$

that is obviously a group under the usual composition, $(\mathrm{Aut}(G), \circ)$.

The next proposition shows how this group classify the systems $\Phi_\nu : \mathcal{P}(G) \to \mathcal{P}(G)$ under the topological conjugation.

Theorem 10. *Let $\sim$ be a relation on $\mathcal{P}(G)$ given by*

$$\nu_1 \sim \nu_2 \Longleftrightarrow \exists \varphi \in \mathrm{Aut(G)} \ s.t. \ \varphi_\sharp(\nu_1) = \nu_2.$$

Then $\sim$ is an equivalence relation and, for each $\nu \in \mathcal{P}(G)$ the set

$$\{\Phi_{\nu'} \mid \nu' \in [\nu] \in \mathcal{P}(G)/\sim\},$$

is such that any two systems Φ_{ν_1} and Φ_{ν_2} are topologically conjugated.

Proof. In order to see that $\sim$ as defined above is a equivalence relation, we just observe that it is consequence that $Id \in \mathrm{Aut}(G)$, every $\varphi \in \mathrm{Aut}(G)$ has an inverse in $\mathrm{Aut}(G)$ and $\mathrm{Aut}(G)$ is closed under composition, so $\sim$ is reflexive, symmetric and transitive.

On the other hand, given $\nu \in \mathcal{P}(G)$ we apply Lemma 9 to show that $\Phi_{\nu'}$ is topologically conjugated to Φ_ν for each $\nu' \in [\nu]$.

□

We remark that if $\Phi_{\nu'}$ is topologically conjugated to Φ_ν we can not claim that $[\nu'] \neq [\nu]$.

2.2 Limit sets

Here we study asymptotic dynamical properties of the convolution map.

Definition 11. *Let $T : X \to X$ a continuous map. A point $x \in X$ is to be said recurrent if there exists a sequence $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ with $\lim_{k\to+\infty} n_k = +\infty$, such that $\lim_{k\to+\infty} T^{n_k}(x) = x$.*

Example 12. *Consider $\nu = \delta_g$, then given $\mu \in \mathcal{P}(G)$, we have that $\Phi_\nu^n(\mu) = (L_{g^n})_\sharp\mu$. Hence, if μ is a non-wandering point then $(L_{g^n})_\sharp\mu \to \mu$. If we take the two dimensional torus $G = \mathbb{T}^2$ and $\mu = \delta_{(x,y)}$, we can see that $(x, y)^n = (nx \text{ mod } 1, ny \text{ mod } 1)$. So if we consider $\nu = \delta_{(\frac{1}{3},\frac{1}{2})}$, then the sequence $\{\Phi_\nu^{6k}(\mu)\}_{k\in\mathbb{N}}$ converges to μ, and it implies that μ is a recurrent point.*

We also define the ω-limit set.

Definition 13. *Let $T : X \to X$ be a continuous map. Let $x \in X$. A point $y \in X$ is in the ω-limit of x if there exists a sequence of natural numbers $n_k \to \infty$ (as $k \to \infty$) such that $T^{n_k}(x) \to y$.*

Is usual in Probability Theory, to study the powers of the convolution, that is, the convergence of

$$\{\nu^n \,|\, n \in \mathbb{N}\},$$

and its accumulation points $\Gamma_\nu = \overline{\{\nu^n \,|\, n \in \mathbb{N}\}}$. For instance in [12], the author considered μ a probability measure on $d \times d$ non-negative matrices, S_ν the support of ν and he proved a result about the convergence of the sequence $\{\nu^n\}$ based in the concept of cyclicity, which is defined as follows: let $A_1, ..., A_k$ be pairwise disjoint subset of $\{1, ..., d\}$, then S_ν is cyclic with respect to $A_1, ..., A_k$ if for each $x \in S_\nu$, with $A_{k+i} = A_i$, $1 \le i \le k$ we have

$$\sum_{j\in A_{m+1}} x_{ij} > 0, \quad \sum_{j\in A_{m+1}^c} x_{ij} = 0, \; i \in A_m, \; 1 \le k \le m.$$

Theorem 14 (See [12])**.** *Let ν be a probability measure on $d \times d$ non-negative matrices and S_ν the support of ν. Denote by S the semigroup generated by S_ν. Let us assume that the sequence $\{\nu^n\}$ is tight and the minimal rank of the matrices in S is 2. Let us further assume that none of the matrices is S has a zero row. Then the sequence $\{\nu^n\}$ does not converge if and only if S_ν is cyclic.*

Proposition 15. *Given $g \in G$, we have that $\omega(\delta_g) = (R_g)_\sharp(\Gamma_\nu)$.*

Proof. We know, by lemma 3, that $\Phi_\nu(\delta_g) = (R_g)_\sharp\nu$. On the other hand if

$\bar{x} = (x_1, ..., x_{n-1})$, we have that

$$\begin{aligned}\Phi_\nu^{n-1}(\Phi_\nu(\delta_g))(E) &= \int_G \int_G \chi_E(x_1...x_n) d\nu^{n-1}(\bar{x}) d(R_g)_\sharp \nu(x_n) \\ &= \int_G \int_G \chi_E(R_g(x_1...x_n)) d\nu^{n-1}(\bar{x}) d\nu(x_n) \\ &= \int_G \int_G \chi_{(R_g)^{-1}(E)}(x_1...x_{n-1}x_n) d(R_g)_\sharp(\nu)(x) d\nu(y) \\ &= \nu^n(R_g^{-1}(E)) = (R_g)_\sharp(\nu^n)(E).\end{aligned}$$

and it implies that

$$\Phi_\nu^n(\delta_g) = (R_g)_\sharp \nu^n.$$

Then we notice that

$$\bar{\nu} = \lim_{n\to\infty} \Phi_\nu^n(\delta_g) = \lim_{n\to\infty} (R_g)_\sharp \nu^n = (R_g)_\sharp \Big(\lim_{n\to\infty} \nu^n \Big).$$

Hence, if $\nu' \in \omega(\nu)$, then $(R_g)_\sharp \nu' = \bar{\nu} \in \omega(\delta_g)$. □

Corollary 16. $\omega(\delta_e) = (R_e)_\sharp(\Gamma_\nu) = \Gamma_\nu$.

Example 17. *Let us consider $G = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and take $(x, y) \in G$, such that $x \neq 0$ and $\frac{x}{y} \in \mathbb{R}\backslash\mathbb{Q}$. Then we have that the sequence $\{(nx \ mod \ 1 \ , ny \ mod \ 1 \ )\}_{n\in\mathbb{N}}$ is dense in G. It implies that if we take $\nu = (x, y)$, then for all $g \in G$ there exists a sequence $\{n_k\}_{k\in\mathbb{N}}$, with $\lim_{k\to+\infty} n_k = +\infty$, such that $\lim_{k\to+\infty} \Phi_\nu^{n_k}(\mu) = (L_g)_\sharp\mu$. So $(L_g)_\sharp\mu \in \omega(\mu)$, $\forall g \in G$.*

Proposition 18. *Given $\nu \in \mathcal{P}(G)$, for each $\mu \in \mathcal{P}(G)$ we consider the set $\omega(\mu)$, then*

$$\Gamma_\nu * \mu := \{\bar{\nu} * \mu : \bar{\nu} \in \Gamma_\nu\} \subseteq \omega(\mu).$$

Proof. If $\bar{\nu} \in \Gamma_\nu$, then there exists a sequence $\{n_k\}_{k\in\mathbb{N}}$, such that

$$\lim_{k\to\infty} \Phi_\nu^{n_k}(\nu) = \lim_{k\to\infty} \nu^{n_k+1} = \bar{\nu}.$$

Hence, by the continuity of Φ_ν, we have that

$$\lim_{k\to\infty} \Phi_\nu^{n_k+1}(\mu) = \lim_{k\to\infty} \nu^{n_k+1} * \mu = \bar{\nu} * \mu.$$

Then we see that $\bar{\nu} * \mu \in \omega(\mu)$. □

In the next we will specify this results for finite abelian groups, see Theorem 38.

3 Topological Entropy

Here we are interested in study the topological entropy of the map Φ_ν. We start it with a very interesting result.

Lemma 19. *Let G be a group, $\nu = \sum_{i=1}^n a_i\delta_{g_i}$ and $\mu, \mu' \in \mathcal{P}(G)$, we have that*

$$d(\Phi_\nu(\mu), \Phi_\nu(\mu')) \leq \sup_i d((L_{g_i})_\sharp\mu, (L_{g_i})_\sharp\mu'),$$

where d represents the weak-$*$ distance.

Proof. We notice that

$$\begin{aligned}
d(\Phi_\nu(\mu), \Phi_\nu(\mu')) &= \sum_{i=1}^{\infty} \frac{1}{2^i}\Big| \int_G f_i(x)d(\nu * \mu)(x) - \int_G f_i(x)d(\nu * \mu')(x)\Big| \\
&\leq \sum_{i=1}^{\infty} \frac{1}{2^i} \sum_{j=1}^{n} a_j \Big| \int_G f_i(g_jy)d\mu(x) - \int_G f_i(g_jy)d\mu'(x)\Big| \\
&= \sum_{j=1}^{n} a_j \sum_{i=1}^{\infty} \frac{1}{2^i}\Big| \int_G f_i(x)d((L_{g_j})_\sharp\mu)(x) - \int_G f_i(xy)d((L_{g_j})_\sharp\mu')(x)\Big| \\
&\leq \sum_{j=1}^{n} a_j d((L_{g_j})_\sharp\mu, (L_{g_j})_\sharp\mu') \leq \sup_i d((L_{g_i})_\sharp\mu, (L_{g_i})_\sharp\mu'), .
\end{aligned}$$

□

Definition 20. *A distance $d : G \times G \to \mathbb{R}_+$ on topological group G is said right (or left) invariant if $d(xy, zy) = d(x, z)$ (or $d(yx, yz) = d(x, z)$) for all $y \in G$.*

Corollary 21. *Let G be a group, $\nu = \sum_{i=1}^n a_i\delta_{g_i}$, and the distance compatible with the topology is left invariant, then Φ_ν is a non-expansive map.*

Proof. In that case we have that

$$d(\Phi_\nu(\mu), \Phi_\nu(\mu')) \leq \sup_i d((L_{g_i})_\sharp\mu, (L_{g_i})_\sharp\mu') = d(\mu, \mu'),$$

and it implies $d(\Phi_\nu(\mu), \Phi_\nu(\mu')) \leq d(\mu, \mu')$. □

Corollary 22. *Let G be a compact topological group metrizable. If the metric of G is left-invariant, then Φ_ν is non-expansive for any $\nu \in \mathcal{P}(G)$.*

Proof. We only need to notice that the map of convolution is continuous, then given $\nu \in \mathcal{P}(G)$ and $\varepsilon > 0$, there exist $\delta > 0$ and $\nu' \in \mathcal{P}(G)$, such that $\nu = \sum_{i=1}^{n} a_i \delta_{g_i}$, $d(\nu, \nu') < \delta$ and $d(\nu * \mu, \nu' * \mu) < \frac{\varepsilon}{2}$. It implies that

$$\begin{aligned} d(\Phi_\nu(\mu), \Phi_\nu(\mu')) &\leq d(\Phi_{\nu'}(\mu), \Phi_\nu(\mu)) + d(\Phi_{\nu'}(\mu'), \Phi_\nu(\mu)) + d(\Phi_{\nu'}(\mu'), \Phi_\nu(\mu')) \\ &\leq d(\mu, \mu') + 2d(\nu, \nu') \leq d(\mu, \mu') + \varepsilon. \end{aligned}$$

Since the above inequality holds for any $\varepsilon > 0$, the result follows. □

Theorem 23. *If G a compact metrizable group, and the distance compatible with the topology is left invariant, then $h(\Phi_\nu) = 0$, where $h(\Phi_\nu)$ is the topological entropy of Φ_ν.*

Proof. By Corollary 21 Φ_ν is a non expansive map and it implies $h(\Phi_\nu) = 0$. □

For instance, if G is a compact Lie groups, we have that there exists a distance on G, which generates the topology of G and is left-invariant. Then by Theorem 23, we know that Φ_ν has entropy equals zero.

4 Finite Groups

Here we will develop the dynamics of convolution for a finite group. Our main goal is to understand the asymptotic behavior of the convolution map and the classification under topological conjugation. We will always denote by $(G = \{g_0, g_1, ..., g_{n-1}\}, \cdot)$ a finite group of order n where $g_0 = e$ is the neutral element of the operation " $\cdot$ ".
We remind that the space of real continuous functions in G, $C(G, \mathbb{R})$ is identified with $\mathbb{R}^n$ and we denote a function $f \in C(G, \mathbb{R})$ by the row vector

$$f(G) = (f(g_0), f(g_1), ..., f(g_{n-1})) \in \mathbb{R}^n.$$

As usual, the dual of $C(G, \mathbb{R})$ is identified with $(\mathbb{R}^n)^* \simeq \mathbb{R}^n$, is the space of signed measures over G,

$$C(G, \mathbb{R})' = \left\{ \mu = \sum_{i=0}^{n-1} p_i \delta_{g_i},\ p = (p_0, p_1, ..., p_{n-1}) \in \mathbb{R}^n \right\}.$$

in this work we denote

$$\int_G f d\mu = \sum_{i=0}^{n-1} p_i f(g_i) = \langle f(G), p \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product.

In this setting, if $\Delta_n = \{p \in \mathbb{R}^n \mid p_i \in [0,1], \text{ and } \sum_{i=0}^{n-1} p_i = 1\}$ then

$$\mathcal{P}(G) = \left\{ \sum_{i=0}^{n-1} p_i \delta_{g_i} \in C(G, \mathbb{R})' \mid p \in \Delta_n \right\}.$$

If $\nu = \sum_{i=0}^{n-1} p_i \delta_{g_i}$ and $\mu = \sum_{i=0}^{n-1} q_i \delta_{g_i}$ we define the convolution between them as

$$(\nu * \mu)(f) = \int_G f d(\nu * \mu) = \int_G \int_G f(gh) d\nu(g) d\mu(h).$$

Defining $f(G^2)$ as

$$f(G^2) = \begin{bmatrix} f(g_0 g_0) & \cdots & f(g_0 g_{n-1}) \\ \vdots & \ddots & \vdots \\ f(g_{n-1} g_0) & \cdots & f(g_{n-1} g_{n-1}) \end{bmatrix}$$

we get an characterization of the convolution in coordinates.

Theorem 24. *If* $\nu = \sum_{i=0}^{n-1} p_i \delta_{g_i} \simeq p$ *and* $\mu = \sum_{i=0}^{n-1} q_i \delta_{g_i} \simeq q$ *then*

$$(\nu * \mu)(f) = \langle q, f(G^2) \cdot p \rangle.$$

Proof. Indeed, if we write

$$\begin{aligned}(\nu * \mu)(f) &= \int_G \int_G f(gh) d\nu(g) d\mu(h) = \int_G \sum_{i=0}^{n-1} p_i f(g_i h) d\mu(h) \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} q_i p_j f(g_i g_j) = \langle q, f(G^2) \cdot p \rangle.\end{aligned}$$

□

From the Theorem 24 we define the affine map $\Phi_\nu : \mathcal{P}(G) \to \mathcal{P}(G)$, where

$$\Phi_\nu(\mu)(E) = \nu * \mu(E),$$

for any $E \subset G$.

Since, $\mathcal{P}(G)$ is an affine space of codimension 1 in $C(G, \mathbb{R})'$ we know that Φ_ν is given by an bi-stochastic matrix. In order to get the next result

we define a new matrix obtained by a measure $\nu \simeq (p_0, ..., p_{n-1}) \in \mathcal{P}(G)$. If we denote,

$$G^{-1} \times G = \begin{bmatrix} g_0^{-1} g_0 & \cdots & g_0^{-1} g_{n-1} \\ \vdots & \ddots & \vdots \\ g_0^{-1} g_{n-1} & \cdots & g_{n-1}^{-1} g_{n-1} \end{bmatrix},$$

then

$$\nu(G^{-1} \times G) = \begin{bmatrix} \nu(g_0^{-1} g_0) & \cdots & \nu(g_0^{-1} g_{n-1}) \\ \vdots & \ddots & \vdots \\ \nu(g_0^{-1} g_{n-1}) & \cdots & \nu(g_{n-1}^{-1} g_{n-1}) \end{bmatrix},$$

where $\nu(g_i^{-1} * g_j) = p_m$ if $g_i^{-1} * g_j = g_m$.

Lemma 25. *Given $\nu, \mu \in \mathcal{P}(G)$, we can represent $\Phi_\nu(\mu)$ as*

$$\Phi_\nu(\mu) = \nu * \mu = \mu \cdot \nu(G^{-1} \times G).$$

Proof. If $\nu = \sum_{i=0}^{n-1} p_i \delta_{g_i}$ and $\mu = \sum_{i=0}^{n-1} q_i \delta_{g_i}$ we set $\Phi_\nu(\mu) = \nu * \mu = \sum_{k=0}^{n-1} a_k \delta_{g_k}$.

From the Theorem 24 we know that,

$$a_k = \sum_{g_i g_j = g_k} p_i q_j = \sum_{i=0}^{n-1} \{q_i p_j \mid g_j = g_i^{-1} g_k\}.$$

Since the equation $g_i g_j = g_k$ has an unique solution for a fixed k and for each i we have $j(i,k)$ well determined. It allow us to write,

$$a_k = q_0 \cdot p_{j(0,k)} + ... + q_{n-1} \cdot p_{j(n-1,k)}.$$

Using matrices we have

$$\begin{bmatrix} a_0 & \cdots & a_{n-1} \end{bmatrix} = \begin{bmatrix} q_0 & \cdots & q_{n-1} \end{bmatrix} \cdot \begin{bmatrix} p_{j(0,0)} & \cdots & p_{j(n-1,0)} \\ \vdots & \ddots & \vdots \\ p_{j(0,n-1)} & \cdots & p_{j(n-1,n-1)} \end{bmatrix}.$$

and we get the first formula

$$\Phi_\nu(\mu) = \nu * \mu = \mu \cdot \nu(G^{-1} \times G).$$

□

Thus, if we desire to compute the iterates of $\Phi_\nu(\mu)$ we have

$$\Phi_\nu^m(\mu) = \mu \cdot \nu(G^{-1} \times G)^m,$$

so we can estimate the long time behavior of Φ_ν from the powers of the matrix $\nu(G^{-1} \times G)$.

Example 26. *We consider $G = (\mathbb{Z}_3, +)$, so G^2 is given by*

$$G^2 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} \text{ and } G^{-1} \times G = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

It is easy to check that

$$f(G^2) = \begin{bmatrix} f(0) & f(1) & f(2) \\ f(1) & f(2) & f(0) \\ f(2) & f(0) & f(1) \end{bmatrix}, \text{ and } \nu(G^{-1} \times G) = \begin{bmatrix} q_0 & q_1 & q_2 \\ q_2 & q_0 & q_1 \\ q_1 & q_2 & q_0 \end{bmatrix}.$$

Thus if $\mu \simeq p$ and $\nu \simeq q$

$$\langle p, \begin{bmatrix} f(0) & f(1) & f(2) \\ f(1) & f(2) & f(0) \\ f(2) & f(0) & f(1) \end{bmatrix} q\rangle = \Phi_\nu(\mu)(f) = \langle p \cdot \begin{bmatrix} q_0 & q_1 & q_2 \\ q_2 & q_0 & q_1 \\ q_1 & q_2 & q_0 \end{bmatrix}, f(G)\rangle.$$

In this case, if $\nu = (1/3, 1/4, 5/12)$ then

$$\nu(G^{-1} \times G) = \begin{bmatrix} 1/3 & 1/4 & 5/12 \\ 5/12 & 1/3 & 1/4 \\ 1/4 & 5/12 & 1/3 \end{bmatrix}$$

As in $\mathbb{R}^n$ all norms are equivalents, we will consider on $\mathbb{R}^n$ the norm given by

$$\|q\|_\infty = \sum_{i=1}^{n} |q_i|,$$

with $\nu \simeq q = (q_0, ..., q_{n-1})$.

Lemma 27. *If $\nu \in \mathcal{P}(G)$, then Φ_ν is an isometry.*

Proof. As $\Phi_\nu(\mu)$ is a probability, then $\|\Phi_\nu(\mu\| = 1 = \|\mu\|$. $\square$

Remark 28. *We know, by the previous section, that if there exist an isomorphism $\varphi : G \to G$, then its push forward is a topological conjugation between Φ_{ν_1} and Φ_{ν_2}, since $\varphi_\sharp(\nu_1) = \nu_2$. As φ is a bijection on*

$G = \{g_0, ..., g_{n-1}\}$, it induces a permutation on the set $\{0, 1, ..., n-1\}$, which we will denote by

$$\sigma : \{0, 1, ..., n-1\} \to \{0, 1, ..., n-1\}.$$

Then we have that if $\nu = (p_1, ..., p_{n-1})$, then

$$\varphi_\sharp(\nu) = (p_{\sigma^{-1}(0)}, ..., p_{\sigma^{-1}(n-1)}).$$

It implies that if $\nu_1 \sim \nu_2$, then ν_2 is a redistribution of the weights of the measure ν_1. So, given $\nu \in \mathcal{P}(G)$, the cardinal of the set $[\nu]$ is at most $n!$. Hence the cardinal of the set $\{\Phi_{\nu'} : \nu' \in [\nu]\}$, that is, the number of distinct classes of conjugation, is at most $n!$.

Definition 29. *A stochastic matrix $A = (a_{ij})$ is called semi- positive if there is $N \in \mathbb{N}$ such that all the entries of the matrix A^N are positive.*

Definition 30. *A matrix A with positive entries is called doubly-stochastic if its rows and columns sum 1.*

From [11] we get the theorem that establish the convergence of doubly stochastic matrices.

Theorem 31. *If A is $n \times n$ and semi-positive doubly stochastic matrix, then*

$$\lim_{m\to\infty} A^m = \frac{1}{n} J,$$

where $J = (a_{ij})$, $a_{ij} = 1$ for all i, j.

We will apply Theorem 31 to get a interesting result about the orbits of the map Φ_ν.

Definition 32. *Let G be a finite abelian group of order n. We say that G is finitely generated if there exist $g_1, ..., g_k \in G$ such that for all $g \in G$ we have that $g = g_1^{r_1} \cdots g_k^{r_k}$, with $r_j \in \{0, 1, ..., n\}$.*

We remember the definition of the support of a given measure. Let G a finite group and $\nu = (p_0, ..., p_{n-1}) \in \mathcal{P}(G)$. The support of ν is the set

$$\mathrm{supp}(\nu) = \{g_i \in G : \nu(g_i) = p_i > 0\}.$$

We will denote by H the subgroup of G generated by $\mathrm{supp}(\nu)$, i.e., $H = \langle \mathrm{supp}(\nu) \rangle$. In order to get the next result we need a new definition and a Lemma. We start with the definition:

Definition 33. *(Acyclic) Given $\nu = p \in \mathcal{P}(G)$, we define the set $Z_+(p)^m$ by*

$$Z_+(p)^m = \{g_{i_1}...g_{i_m} : g_{i_k} \in supp(\nu)\}.$$

Let H the subgroup of G generated by $supp(\nu)$. We say that ν is a acyclic probability measure if there exist $N \in \mathbb{N}$ such that $Z_+(p)^N = H$. In particular, $Z_+(p)^1 = supp(\nu)$.

Example 34. *(a) Let $g \in G$ an element of order 2 and $\nu = \delta_g$. In this case $H = \{e, g\}$ and*

$$Z_+(p)^m = \begin{cases} e, & \text{if } m \text{ is even} \\ g, & \text{if } m \text{ is odd.} \end{cases}$$

From it follows that ν is not acyclic probability measure.
(b) Let H a cyclic group generated by g and $\nu = \alpha\delta_e + (1-\alpha)\delta_g$, $0 < \alpha < 1$. Then ν is a acyclic. In fact, if $H = \{e, g, ..., g^{n-1}\}$, then

$$Z_+(p)^n = \{e^n, e^{n-1}g, e^{n-2}g^2, ...e^1g^{n-1}\} = H.$$

Example 35. *Let G be a finite abelian group of order n and $\nu \in \mathcal{P}(G)$. We can make the identification $\nu = \sum_i p_i\delta_{g_i} \simeq p = (p_0, ..., p_{n-1})$. If $Z_+(p) = \{g, h\}$ and $H = \langle g^{-1}h\rangle$, then ν is a acyclic. In fact, to see it we only need to notice that*

$$g^{n-k}h^k = (g^{-1}h)^k.$$

Example 36. *Let $H = \langle g_1, ..., g_k\rangle$ be a finitely generated abelian subgroup of G and $\nu \in \mathcal{P}(G)$. If $\nu = p \in \mathcal{P}(G)$ is such*

$$Z_+(p) = \{e, g_1, ..., g_k\}$$

ν is a acyclic.

Remark 37. *If we have that $|G| = n$ and the support of ν has more than $\dfrac{n}{2} + 1$ elements, then ν is acyclic, in particular $\nu(G^{-1} \times G)$ semi-positive.*

When the probability ν is acyclic we have the following theorem:

Theorem 38. *Let $G = \{g_0, ..., g_{n-1}\}$ a finite group, $\nu = p \in \mathcal{P}(G)$ acyclic and $H = \langle Z_+(p)\rangle$. The matrix $p(G^{-1} \times G) = (p(g_i^{-1}g_j))_{i,j}$ satisfies $\lim_{n\to\infty} p(G^{-1} \times G)^n = B$, where B is the matrix given by*

$$\begin{pmatrix} \frac{1}{|H|}J & 0 & \dots & 0 \\ 0 & \frac{1}{|H|}J & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \frac{1}{|H|}J \end{pmatrix},$$

where 0 *is the null matrix of order* $|H|$ *and* J *is the matrix of order* $|H|$*, with all the coefficients equal to 1.*

Proof. The prove of this result follows form the lemmas below. □

Remark 39. *Let us consider an acyclic probability* $\nu = p \in \mathcal{P}(G)$*, and the subgroup* H *generated by* $Z_+(p)$*. We suppose that* $|H| = n$*. We can take the equivalence classes determined by* H *in* G*, i.e,*

$$gH = \{gh : h \in H\}.$$

We know that G *can be written as a disjoint union of the equivalence classes determined by* H*, and as* G *is finite* H *is also a finite group. Then we can write* G *as follows*

$$\begin{aligned} G &= \{e, h_1, ..., h_k, g_1h_1, ..., g_1, g_1h_k, g_2h_1, ..., g_2, g_2h_k, ..., g_l, g_lh_1, ..., g_lh_k\} \\ &= \{H, g_1H, ..., g_lH\}, \end{aligned}$$

where $g_iH \cap g_jH = \emptyset$ *for* $i \neq j$*. Then we have that the matrix* $A = p(G^{-1} \times G)$ *is given by*

$$A = \begin{pmatrix} p(H^{-1} \times H) & p(H^{-1} \times g_1H) & \dots & p(H^{-1} \times g_lH) \\ p(H^{-1}g_1^{-1} \times H) & p(H^{-1}g_1^{-1} \times g_1H) & \dots & p(H^{-1}g_1^{-1} \times g_lH) \\ \vdots & \vdots & \vdots & \vdots \\ p(H^{-1}g_l^{-1} \times H) & \dots & \dots & p(H^{-1}g_l^{-1} \times g_lH) \end{pmatrix},$$

where the blocks in the diagonal are always the matrix $p(H^{-1} \times H)$*.*

Lemma 40. *The blocks* $p(H^{-1}g_i^{-1} \times g_jH)$*,* $p(H^{-1}g_i^{-1} \times H)$ *and* $p(H^{-1} \times g_jH)$ *are always the null matrix for* $i \neq j$*.*

Proof. Take the block $p(H^{-1}g_1^{-1} \times g_2H)$ and notice that

$$\begin{aligned} p((h_i^{-1}g_1^{-1})(g_2h_j)) > 0 &\Leftrightarrow (h_i^{-1}g_1^{-1})(g_2h_j) \in Z_+(p) \subset H \\ &\Rightarrow g_1^{-1}g_2 \in H \Rightarrow g_1H = g_2H, \end{aligned}$$

but it is a contradiction, since $g_1H \cap g_2H = \emptyset$. By analogous computations we have the result for the others cases. □

By Lemma 40, we have that the powers of the matrix $p(G^{-1} \times G)$ are given by

$$p(G^{-1} \times G)^n = \begin{pmatrix} p(H^{-1} \times H)^n & 0 & \dots & 0 \\ 0 & p(H^{-1} \times H)^n & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & p(H^{-1} \times H)^n \end{pmatrix}.$$

Lemma 41. *The matrix $p(H^{-1}\times H)$ is semi-positive.*

Proof. Let us consider the matrix $A=(a_{ij})_{i,j}:=(p(h_i^{-1}h_j))_{i,j}$. Then we notice that

$$\begin{aligned} a_{ij}>0 &\Leftrightarrow h_i^{-1}h_j\in Z_+(p)\\ &\Leftrightarrow \exists \bar{h}\in Z_+(p) \text{ such that } h_i^{-1}h_j=\bar{h}\\ &\Leftrightarrow h_j=h_i\bar{h},\ \bar{h}\in Z_+(p). \end{aligned}$$

It implies that $a_{ij}>0$ if and only if $h_j\in L_{h_i}(Z_+(p))$. As L_{h_i} is a bijection, in each line we have $|Z_+(p)|$ positive coefficients. Consider now A^2, which we will denote by $A^2=(a^2_{ij})_{i,j}$. Then we have that

$$\begin{aligned} a^2_{ij}>0 &\Leftrightarrow \sum_{k=0}^{n-1} p(h_i^{-1}h_k)p(h_k^{-1}h_j)\\ &\Leftrightarrow \exists k\in\{0,...,n-1\} \text{ such that } p(h_i^{-1}h_k)p(h_k^{-1}h_j)>0\\ &\Leftrightarrow p(h_i^{-1}h_k)>0 \text{ and } p(h_k^{-1}h_j)>0\\ &\Leftrightarrow \exists h',h''\in Z_+(p) \text{ such that } h_k=h_ih',\ h_j=h_kh''\\ &\Leftrightarrow h_j=h_ih'h''\\ &\Leftrightarrow h_j\in L_{h_i}(Z_+(p)^2). \end{aligned}$$

Again, we can see that A^2 has $|Z_+(p)^2|$ positive coefficients. Following by induction, if $A^n=(a^n_{ij})_{i,j}$, then

$$a^n_{ij}>0\Leftrightarrow h_j\in L_{h_i}(Z_+(p)^n).$$

As ν is acyclic we have, from Definition 33 that there exists $N\in\mathbb{N}$, such that for $n>N$

$$a^n_{ij}>0\Leftrightarrow h_j\in L_{h_i}(Z_+(p)^n)=h_iH=H.$$

It implies that for $n>N$ the matrix $A^n=(a^n_{ij})_{i,j}$ has $|H|$ coefficients positive in each line. As the matrix A has order $|H|$ we see that A is semi-positive. □

Lemma 42. *Let $\mu,\nu\in\mathcal{P}(G)$ and σ a permutation on G. Then we have that*

$$\mu\cdot\nu((\sigma(G))^{-1}\times\sigma(G))=\mu\cdot\nu(G^{-1}\times G).$$

Proof. We notice that the convolution does not depend of the order of the group, then

$$\mu\cdot\nu((\sigma(G))^{-1}\times\sigma(G))=\mu*\nu=\mu*\nu(G^{-1}\times G).$$

□

We like to observe that, B is also doubly stochastic and always has 1 as an eigenvalue.

Remark 43. *The main fact used in the Lemma 42 was the fact that the integral does not change under permutation of the group G.*

Using Lemma 42 and Theorem 38, and making some permutation of the elements one can easily conclude that,

Theorem 44. *Let $G = \{g_0, ..., g_{n-1}\}$ a finite group, $\nu = p \in \mathcal{P}(G)$ a acyclic probability and $H = \langle Z_+(p)\rangle$. The matrix $p(G^{-1} \times G) = (p(g_i^{-1} g_j))_{i,j}$ satisfies $\lim_{n\to\infty} p(G^{-1} \times G)^n = B$, where B is the matrix given by*

$$b_{ij} = \begin{cases} 0, & \text{if } g_i^{-1} g_j \notin H \\ \frac{1}{|H|}, & \text{if } g_i^{-1} g_j \in H \end{cases}$$

Example 45. *Take $\bar{G}$ a finite abelian group and $a, b \in \bar{G}$ such that $a^2 = e$ $b^3 = e$ and $\nu = p = \alpha\delta_e + (1-\alpha)\delta_b$, $0 < \alpha < 1$ and $G = \{e, a, b^2, ab, b, ab^2\}$. So we have*

$$\nu(G^{-1}\times G) = \begin{pmatrix} \alpha & 0 & 0 & 0 & (1-\alpha) & 0 \\ 0 & \alpha & 0 & (1-\alpha) & 0 & 0 \\ (1-\alpha) & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & (1-\alpha) \\ 0 & 0 & (1-\alpha) & 0 & \alpha & 0 \\ 0 & (1-\alpha) & 0 & 0 & 0 & \alpha \end{pmatrix}.$$

In that case $Z_+(p) = \{e, b\}$ and $\langle Z_+(p)\rangle = \{e, b, b^2\}$, and by Theorem 44 we have that

$$\lim_{n\to\infty} \nu(G^{-1} \times G)^n = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}.$$

Then given $\mu = (q_0, ..., q_5)$,

$$\lim_{n\to\infty} \Phi_\nu^n(\mu) = \frac{1}{3}\Big(q_0+q_2+q_4, q_1+q_3+q_5, q_0+q_2+q_4, q_+q_3+q_5, q_0+q_2+q_4, q_1+q_3+q_5\Big).$$

4.1 Analysis of the ω-limit set

Initially, we like to observe that the *acyclic* property is dense in $\mathcal{P}(G)$.

Theorem 46. *Let $\nu_0 \in \mathcal{P}(G)$, where G is a finite group. Given $\varepsilon > 0$, there exists $\bar{\nu} \in \mathcal{P}(G)$ such that $\bar{\nu}$ is a acyclic and $d(\bar{\nu}, \nu_0) < \varepsilon$, i.e., the set of acyclic probabilities is dense in $\mathcal{P}(G)$.*

Proof. Let $\varepsilon > 0$ and $\nu_0 = p = \sum_{i=0}^{k-1} p_i \delta_{h_i}$ with

$$Z_+(p) = \{g \in G : \nu(g) > 0\} \text{ and } H = \langle Z_+(p) \rangle = \{h_0, ..., h_{k-1}\}.$$

Then we define $a = \min\{p_i : p_i > 0\}$ and $\bar{\varepsilon} = \dfrac{1}{2}\min\{\varepsilon, a\}$. So we consider the measure $\bar{\nu} = \bar{p} = \sum_{i=0}^{k-1} \bar{p}_i \delta_{h_i}$, where

$$\bar{p}_i = \begin{cases} \frac{\bar{\varepsilon}}{k - |Z_+(p)|}, & \text{if } p_i = 0 \\ p_i - \frac{\bar{\varepsilon}}{|Z_+(p)|}, & \text{if } p_i > 0. \end{cases}$$

Obviously $\bar{\nu} \in \mathcal{P}(G)$ and as

$$d(\nu_0, \bar{\nu}) = \sum_i |p_i - \bar{p}_i| = \sum_{p_i = 0} \frac{\bar{\varepsilon}}{k - |Z_+(p)|} + \sum_{p_i > 0} \frac{\bar{\varepsilon}}{|Z_+(p)|} = 2\bar{\varepsilon} < \varepsilon,$$

so we get the result. $\square$

Remark 47. *From the proof of the theorem above one can see that, the value of the limit of the entries of the matrix is unstable, that is, close to a same measure of probability, we can have probabilities converging to several values, under the perturbation of different weights of the probability. Moreover, the maximum value is an open property. Indeed, by our theorem, the matrix limit B, converges to blocks $\frac{1}{|H|}J$, thus if we increase $Z_+(p)$ we can just decrease asymptotic value $\frac{1}{|H|}$.*

Using Theorem 38, we will try to find conditions for two measures have the same $\omega-$limit, where $\nu = \sum_i p_i \delta_{g_i}$ is a acyclic. First we observe that if $\mu = \sum_i q_i \delta_{g_i} \in \mathcal{P}(G)$, then $\omega(\mu) = \{\mu \cdot B\}$. If we identify μ with the vector $q = \sum_i q_i e_i$ in $\mathbb{R}^n$, where $\{e_i\}_{0 \le i \le n-1}$ is the canonical basis of $\mathbb{R}^n$, we have that

$$q \cdot B = \Big(\sum_i q_i e_i\Big) \cdot B = \sum_i q_i \Big(e_i \cdot B\Big).$$

It implies that $\omega(\mu) = \sum_i q_i\omega(\delta_{g_i})$. So, to determine the $\omega-$limit of a measure it is enough to determine the $\omega-$limit of the measures δ_{g_i}, for all $g_i \in G$. Then we notice that if $H = \langle Z_+(p)\rangle$, $|H| = k$, $|G| = |H|l$, $\bar{\mu} = (q_0, ..., q_{n-1})$, $\mu = \delta_{g_0}$, and if we write

$$\alpha_0 = \sum_{i=0}^{k-1} q_i,\ \alpha_1 = \sum_{i=k}^{2k-1} q_i, ...,\ \alpha_l = \sum_{i=n-k-1}^{n-1} q_i$$

$$\begin{aligned}
&\mu \cdot B = \bar{\mu} \cdot B \\
&\Leftrightarrow \Big(\underbrace{\frac{1}{k}, \frac{1}{k}, ..., \frac{1}{k}}_{k}, 0, ..., 0\Big) = \Big(\underbrace{\frac{1}{k}\alpha_0, ..., \frac{1}{k}\alpha_0}_{k}, \underbrace{\frac{1}{k}\alpha_1, ..., \frac{1}{k}\alpha_1}_{k}, ..., \underbrace{\frac{1}{k}\alpha_l, ..., \frac{1}{k}\alpha_l}_{k}\Big) \\
&\Leftrightarrow \sum_{i=0}^{k-1} q_i = 1,\ \sum_{i=k}^{2k-1} q_i = 0,\ ....,\ \sum_{i=n-k-1}^{n-1} q_i = 0.
\end{aligned}$$

It implies that $\omega(\delta_{g_0}) = \omega(\mu)$, if and only if $\displaystyle\sum_{i=0}^{k-1} q_i = 1$, where $\mu = (1, 0, ..., 0)$. By the same argument used above we can see that

$$\begin{aligned}
&\omega(\delta_{g_i}) = \omega(\delta_{g_0}) \text{ for } 0 \le i \le k-1,\ \omega(\delta_{g_i}) = \omega(\delta_{g_k}) \text{ for } k \le i \le 2k-1, ..., \\
&\omega(\delta_{g_i}) = \omega(\delta_{g_{n-k-1}}) \text{ for } n-k-1 \le i \le n-1,
\end{aligned}$$

and from it follows that $\omega(\mu) = \displaystyle\sum_i q_i\omega(\delta_{g_i}) = \sum_{j=0}^{l} \alpha_j\omega(\delta_{g_{jk}})$, and if $\bar{\mu} = (q_0, ..., q_{n-1})$ and we take $\mu = \delta_{g_i}$, with $mk \le i \le mk-1$,

$$\omega(\bar{\mu}) = \omega(\delta_{g_i}) \Leftrightarrow \alpha_m = \sum_{i=mk}^{mk-1} q_i = 1, \text{ and } \alpha_j = 0 \text{ for } j \ne m.$$

Finally, given $\mu = (q_0, ..., q_{n-1})$ and $\mu' = (q'_0, ..., q'_{n-1})$,

$$\omega(\mu) = \omega(\mu') \Leftrightarrow \sum_{i=0}^{k-1} q_i = \sum_{i=0}^{k-1} q'_i,\ \sum_{i=k}^{2k-1} q_i = \sum_{i=k}^{2k-1} q'_i, ...,\ \sum_{i=n-k-1}^{n-1} q_i = \sum_{i=n-k-1}^{n-1} q'_i.$$

Definition 48. *Let $\nu \in \mathcal{P}(G)$ a acyclic probability measure and $\eta \in \mathcal{P}(G)$. We call the basin of η the set*

$$\{\mu \in \mathcal{P}(G) : \lim_{n\to\infty} \Phi_\nu^n(\mu) = \eta\}.$$

Example 49. *Let's go back to the Example 45 where* $G = \{e, a, b^2, ab, b, ab^2\}$, *in that particular situation,* $\nu = p = \alpha\delta_e + (1-\alpha)\delta_b$, $0 < \alpha < 1$ *and we can rewrite* G *as* $G = \{e, b, b^2, a, ab, , ab^2\}$ *as in Lemma 40.*

Then , given $\mu = (q_0, ..., q_5)$ *and* $\mu' = (q'_0, ..., q'_5)$, *we have*

$$\omega(\mu) = \omega(\mu') \Leftrightarrow \sum_{i=0}^{2} q_i = \sum_{i=0}^{2} q'_i, \text{ and } \sum_{i=3}^{5} q_i = \sum_{i=3}^{5} q'_i.$$

For instance, if $\mu' = (\frac{1}{4}, \frac{1}{2}, 0, \frac{1}{8}, 0, \frac{1}{8})$ *we have*

$$\begin{aligned}\lim_{n\to\infty} \Phi_\nu^n(\mu') &= \frac{1}{3}\Big(q'_0 + q'_1 + q'_2, ..., q'_0 + q'_1 + q'_2, q'_3 + q'_4 + q'_5, ..., q'_3 + q'_4 + q'_5\Big)\\ &= \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right)\\ &= \omega(\mu') = \eta.\end{aligned}$$

So, the basin of attraction of $\eta = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right)$, *that is,*

$$\{\mu = (q_0, ..., q_5) \mid \lim_{n\to\infty} \Phi_\nu^n(\mu) = \eta\}$$

is given by

$$\begin{cases} q_0 + q_1 + q_2 & = \frac{3}{4} \\ q_3 + q_4 + q_5 & = \frac{1}{4} \\ q_0, ..., q_5 & \in [0, 1] \end{cases}$$

that is a convex region of hyperplane in $\mathbb{R}^6$ *of dimension 4, more precisely*

$$\begin{cases} q_0 = \frac{3}{4} - a - b \\ q_1 = a, q_2 = b \\ q_3 = \frac{1}{4} - c - d \\ q_4 = c, q_5 = d \\ a + b \leq \frac{3}{4},\ c + d \leq \frac{1}{4} \\ a, b, c, d \in [0, 1] \end{cases}$$

is the basin of attraction of $\eta = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right)$.

Actually the next theorem shows the this always happens.

Theorem 50. *Let* $\nu = p \in \mathcal{P}(G)$ *a acyclic probability measure and* $H = \langle Z_+(p)\rangle$, *with* $|H| = k$ *and* $|G| = |H|l$. *Given* $\eta \in \mathcal{P}(G)$ *with*

$$\eta = (\underbrace{q_0, ..., q_0}_{k}, \underbrace{q_1, ..., q_1}_{k}, ..., \underbrace{q_{l-1}, ..., q_{l-1}}_{k})$$

The basin of η *is a convex subset of a hyperplane of dimension* $\frac{n(k-1)}{k}$ *in* $\mathbb{R}^n$.

Proof. To prove the convexity of the basin of a given η we only need to notice that if $\mu_1, \mu_2 \in \mathcal{P}(G)$ and $0 \leq \alpha \leq 1$, then

$$\begin{aligned}\Phi_\nu(\alpha\mu_1 + (1-\alpha)\mu_2) &= (\alpha\mu_1 + (1-\alpha)\mu_2) \cdot \nu(G^{-1} \times G) \\ &= \alpha\mu \cdot \nu(G^{-1} \times G) + (1-\alpha)\mu_2 \cdot \nu(G^{-1} \times G).\end{aligned}$$

Hence, if μ_1, μ_2 are in the basin of η, the

$$\begin{aligned}\lim_{n\to\infty} \Phi_\nu^n(\alpha\mu_1 + (1-\alpha)\mu_2) &= \alpha \lim_{n\to\infty} \Phi_\nu^n(\mu_1) + (1-\alpha) \lim_{n\to\infty} \Phi_\nu^n(\mu_2) \\ &= \alpha\eta + (1-\alpha)\eta = \eta.\end{aligned}$$

To prove the second part of the theorem we notice that if $\mu = (q_0', q_1', ..., q_{n-1}')$ is in the basin of η, then $\lim_{n\to\infty} \Phi_\nu^n(\mu) = \eta$ if and only if

$$\begin{cases} \dfrac{1}{k}\displaystyle\sum_{i=0}^{k-1} q_i' = q_0 \\ \quad\vdots \\ \dfrac{1}{k}\displaystyle\sum_{i=n-k-1}^{n-1} q_i' = q_{l-1} \\ q_0, ..., q_{n-1} \in [0,1]. \end{cases}$$

If we forget the restriction $\displaystyle\sum_{i=0}^{n-1} q_i' = 1$ and $q_0, ..., q_{n-1} \in [0,1]$, we have a linear system which has n variables and l linearly independents equations. Then its space of solution is given by an hyperplane of dimension

$$n - l = n - \frac{n}{k} = \frac{n(k-1)}{k}.$$

Then the solution of system above is a convex set given by the intersection of a hyperplane of dimension $\frac{n(k-1)}{k}$ with the simplex $\Delta_n = \{(x_0, ..., x_{n-1}) : \sum_i x_i = 1,\ x_i \in [0,1]\}$. $\square$

Remark 51. *The matrix $A = p(G^{-1} \times G)$ is a circulant Toeplitz matrix if and only if G is a cyclic group. In this case, we can use our previous results to analyze the asymptotic behavior of A^n, that is connected with the convolution in the discrete Fourier transform DFT, see [3] for an excellent survey on Toeplitz matrices. There is a big amount of papers on this subject, that is, the asymptotic behavior of the powers of the convolution in groups and semigroups. In [5] the reader can find some results about this subject .*

Remark 52. *When the group G is locally finite when use the results that we have obtained in the previous section to see what happens with the orbits of the map Φ_ν.*

Definition 53. *We say that a group G is locally finite if any finitely generated subgroup of G is finite.*

Lemma 54. *If G is locally finite, $\nu = \sum_{i=1}^{n} a_i \delta_{g_i}$ and $\mu = \sum_{i=1}^{m} b_i \delta_{h_i}$, then*

$$\Phi_\nu(\mu) = \mu \cdot \nu(G^{-1}_{(\nu,\ \mu)} \times G_{(\nu,\ \mu)}),$$

where $G_{(\nu,\ \mu)}$ is the group generated by the set $\{g_i, h_j\}_{i,j}$

Proof. As G is locally finite, we have that the subgroup generated by the set $\{g_i, h_j\}_{i,j}$ is finite. Let $G_{(\nu,\ \mu)} = \{r_0, ..., r_{k-1}\}$ that subgroup. If we rewrite ν and ν as $\nu = \sum_{i=0}^{k-1} a'_i \delta_{r_i}$ and $\mu = \sum_{i=0}^{k-1} b'_i \delta_{r_i}$, by Lemma 25, we have that

$$\Phi_\nu(\mu) = \mu \cdot \nu(G^{-1}_{(\nu,\ \mu)} \times G_{(\nu,\ \mu)}).$$

□

By Corollary 21, we know that if there exists a metric on G which generates its topology and this metric is left invariant, then $d(\Phi_\nu(\mu), \Phi_\nu(\mu')) \leq d(\mu, \mu')$. We also know that the set of probabilities with a finite number of points in their supports is dense in $\mathcal{P}(X)$. Hence, if we wish to study the behavior of a probability when ν has finite support, we start by analysing the orbit of an element of finite support. It can be summarized in

Proposition 55. *Let G a locally finite group and $\nu \in \mathcal{P}(G)$ has a finite number of points in its support. Given $\mu \in \mathcal{P}(G)$ and $\varepsilon > 0$, there exists $\mu' \in \mathcal{P}(G)$ such that $d(\mu, \mu') < \varepsilon$ and*

$$\Phi^n_\nu(\mu) \approx_\varepsilon \mu' \cdot \nu(G^{-1}_{(\nu,\ \mu')} \times G_{(\nu,\ \mu')})^n,$$

where the symbol $\approx_\varepsilon$ means that $d(\Phi^n_\nu(\mu), \mu' \cdot \nu(G^{-1}_{(\nu,\ \mu')} \times G_{(\nu,\ \mu')})^n) < \varepsilon$.

When ν has a finite number of points in its support and G is locally finite, when can use the Proposition 55 to study the ω-limit set of any probability $\mu \in \mathcal{P}(G)$. Moreover, this analysis can make use of the facts proved in the previous sections, since we know the behavior of the sequence $\{\mu' \cdot \nu(G^{-1}_{(\nu,\ \mu')} \times G_{(\nu,\ \mu')})^n)\}_{n\in\mathbb{N}}$. When the support of ν is an infinite set

we can approximate ν by a probability ν' with a finite number of points in its support. If the metric of G is bi-invariant we have that

$$d(\Phi^n_\nu(\mu), \Phi^n_{\nu'}(\mu')) \leq nd(\nu, \nu') + d(\mu, \mu').$$

The last inequality tells us that is possible to have a good description of the orbit of a probability μ, in each iterate of Φ_ν, in terms of probabilities with a finite number of points in the support.

Email:
baravi@mat.ufrgs.br
oliveira.elismar@gmail.com
patropy@hotmail.com

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